

WARPED PRODUCT SKEW SEMI-INVARIANT SUBMANIFOLDS OF ORDER 1 OF A LOCALLY PRODUCT RIEMANNIAN MANIFOLD

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ABSTRACT. We introduce warped product skew semi-invariant submanifolds of order 1 of a locally product Riemannian manifold. We give a necessary and sufficient condition for skew semi-invariant submanifold of order 1 to be a locally warped product. We also prove that the invariant distribution which is involved in the definition of the submanifold is integrable under some restrictions. Moreover, we find an inequality between the warping function and the squared norm of the second fundamental form for such submanifolds. Equality case is also discussed.

1. INTRODUCTION

The theory of submanifolds is one of the most popular research area in differential geometry. In an almost Hermitian manifold, its almost complex structure determines several types of submanifolds. For example, holomorphic (invariant) submanifolds and totally real (anti-invariant) submanifolds are determined by the behavior of the almost complex structure. In the first case the tangent space of the submanifolds is invariant under the action of the almost complex structure. In the second case the tangent space of the submanifolds is anti-invariant, that is, it is mapped into the normal space. A. Bejancu [4] introduced the notion of CR-submanifolds of a Kählerian manifold as a natural generalization of invariant and anti-invariant submanifolds. A CR-submanifold is said to be proper if it is neither invariant nor anti-invariant. The theory of CR-submanifolds has been a most interesting topics since then. Slant submanifolds are another generalization of invariant and anti-invariant submanifolds. This type submanifolds is defined by B.Y. Chen [9]. Since then such submanifolds have been studied by many geometers (see [3, 8, 17] and references therein). If a slant submanifold is neither invariant nor anti-invariant then it is said to be proper. We observe that a proper CR-submanifold is never a slant submanifold. In [18], N. Papaghiuc introduced the notion of semi-slant submanifolds obtaining CR-submanifolds and slant submanifolds as special cases. A. Carriazo [8], introduced bi-slant submanifolds which is a generalization of semi-slant submanifolds. One of the classes of such submanifolds is that of anti-slant submanifolds. This type submanifolds are also generalization of slant and CR-submanifolds. However, B. Şahin [23] called these submanifolds as hemi-slant submanifolds because of that the name anti-slant seems to refer that it has no slant factor. He also observed that there is no inclusion between proper

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hemi-slant submanifolds and proper semi-slant submanifolds. We note that hemi-slant submanifolds are also studied under the name of pseudo-slant submanifolds (see [14, 27]).

Skew CR-submanifolds of a Kählerian manifold are first defined by G.S. Ronsse in [19]. Such submanifolds are a generalizations of bi-slant submanifolds. Consequently, invariant, anti-invariant, CR, slant, semi-slant and hemi-slant submanifolds are particular cases of skew CR-submanifolds. We notice that CR-submanifolds in Kählerian manifolds correspond to semi-invariant submanifolds [5] in locally product Riemannian manifolds. Therefore, skew CR-submanifolds in Kählerian manifolds correspond to skew semi-invariant submanifolds in locally product Riemannian manifolds. For the fundamental properties and further studies of skew CR-submanifolds; see [19] and [26]. Skew semi-invariant submanifolds of a locally product Riemannian manifold were studied first by X. Liu and F.-M. Shao in [16].

The notion of warped product was initiated by R.L. Bishop and B. O'Neill [6]. Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively. Let f be positive differentiable function on M_1 . The warped product $M = M_1 \times_f M_2$ of M_1 and M_2 is the Riemannian manifold $(M_1 \times M_2, g)$, where

$$g = g_1 + f^2 g_2 .$$

More explicitly, if $U \in T_p M$, then

$$\|U\|^2 = \|d\pi_1(U)\|^2 + (f^2 \circ \pi_1) \|d\pi_2(U)\|^2 ,$$

where $\pi_i, i = 1, 2$, are the canonical projections $M_1 \times M_2$ onto M_1 and M_2 respectively. The function f is called the *warping function* of the warped product. If the warping function is constant, then the manifold M is said to be *trivial*. It is also known that M_1 is totally geodesic and M_2 is totally umbilical from [6]. For a warped product $M_1 \times_f M_2$, we denote by \mathcal{D}_1 and \mathcal{D}_2 the distributions given by the vectors tangent to leaves and fibers respectively. Thus, \mathcal{D}_1 is obtained from tangent vectors to M_1 via horizontal lift and \mathcal{D}_2 is obtained by tangent vectors of M_2 via vertical lift. Let U be a vector field on M_1 and V be vector field on M_2 , then from Lemma 7.3 of [6], we have

$$(1.1) \quad \nabla_U V = \nabla_V U = U(\ln f)V ,$$

where ∇ is the Levi-Civita connection on $M_1 \times_f M_2$.

Warped product submanifolds have been studying very actively since B.Y. Chen [10] introduced the notion of CR-warped product in Kählerian manifolds. In fact, different type warped product submanifolds of different kinds structures are studied last thirteen years. For example; see [2, 15, 21, 22, 23, 24, 27]. Most of the studies related to this topic can be found in the survey book [11]. Recently, B. Şahin [24] introduced the notion of skew CR-warped product submanifolds of Kählerian manifolds which is a generalization of different kind warped product submanifolds studied by many authors. We note that warped product skew CR-submanifolds of a cosymplectic manifold were studied in [15].

In this paper, we define and study warped product skew semi-invariant submanifolds of order 1 of a locally product Riemannian manifold. We give an illustrate

example and prove a characterization theorem for the mixed totally geodesic proper skew semi-invariant submanifold using some lemmas. In general, the invariant distribution of a submanifold is not integrable in a locally product Riemannian manifold. However, we prove that the invariant distribution of a warped product skew semi-invariant submanifold of order 1 is integrable in a locally product Riemannian manifold under some restrictions. Finally, we obtain an inequality between the warping function and the squared norm of the second fundamental form for such submanifolds. Equality case is also considered.

2. PRELIMINARIES

Let (\bar{M}, g, F) be a locally product Riemannian manifold or, (briefly, l.p.R. manifold). It means that [28] \bar{M} has a tensor field F of type $(1, 1)$ on \bar{M} such that, $\forall \bar{U}, \bar{V} \in T\bar{M}$, we have

$$(2.1) \quad F^2 = I, (F \neq \pm I), \quad g(F\bar{U}, F\bar{V}) = g(\bar{U}, \bar{V}) \quad \text{and} \quad (\bar{\nabla}_{\bar{U}} F)\bar{V} = 0 ,$$

where g is the Riemannian metric, $\bar{\nabla}$ is the Levi-Civita connection on \bar{M} and I is the identifying operator on the tangent bundle $T\bar{M}$ of \bar{M} .

Let M be a submanifold of a l.p.R. manifold (\bar{M}, g, F) as an isometrically immersed. Let ∇ and ∇^\perp be the induced, and induced normal connection in M and the normal bundle $T^\perp M$ of M , respectively. Then for all $U, V \in TM$ and $\xi \in T^\perp M$ the Gauss and Weingarten formulas are given by

$$(2.2) \quad \bar{\nabla}_U V = \nabla_U V + h(U, V)$$

and

$$(2.3) \quad \bar{\nabla}_U \xi = -A_\xi U + \nabla_U^\perp \xi$$

where h is the *second fundamental form* of M and A_ξ is the Weingarten endomorphism associated with ξ . The second fundamental form h and the *shape operator* A related by

$$(2.4) \quad g(h(U, V), \xi) = g(A_\xi U, V) .$$

The *mean curvature vector field* H is given by $H = \frac{1}{m}(\text{trace } h)$, where $\dim(M) = m$. The submanifold M is called *totally geodesic* in \bar{M} if $h = 0$ and *minimal* if $H = 0$. If $h(U, V) = g(U, V)H$ for all $U, V \in TM$, then M is *totally umbilical*.

3. SKEW SEMI-INVARIANT SUBMANIFOLDS OF ORDER 1 OF A LOCALLY PRODUCT RIEMANNIAN MANIFOLD

Let \bar{M} be a l.p.R. manifold with Riemannian metric g and almost product structure F . Let M be Riemannian submanifold isometrically immersed in \bar{M} . For any $U \in TM$, we write

$$(3.1) \quad FU = TU + NU .$$

Here TU is the tangential part of FU , and NU is the normal part of FU . Similarly, for any $\xi \in T^\perp M$, we put

$$(3.2) \quad F\xi = t\xi + \omega\xi ,$$

where $t\xi$ is the tangential part of $F\xi$, and $\omega\xi$ is the normal part of $F\xi$.

Using (2.1) and (3.1), we have $g(T^2U, V) = g(T^2V, U)$ for all $U, V \in TM$. It says that T^2 is a symmetric operator on the tangent space $T_pM, p \in M$. Therefore its eigenvalues are real and diagonalizable. Moreover, its eigenvalues are bounded by 0 and 1. For each $p \in M$, we set

$$\mathcal{D}_p^\lambda = \text{Ker}\{T^2 - \lambda^2(p)I\}_p,$$

where I is the identity endomorphism and $\lambda(p)$ belongs to closed interval $[0, 1]$ such that $\lambda^2(p)$ is an eigenvalue of T_p^2 . Since T_p^2 is symmetric and diagonalizable, there is some integer k such that $\lambda_1^2(p), \dots, \lambda_k^2(p)$ are distinct eigenvalues of T_p^2 and T_pM can be decomposed as a direct sum of mutually orthogonal eigenspaces, i.e.

$$T_pM = \mathcal{D}_p^{\lambda_1} \oplus \dots \oplus \mathcal{D}_p^{\lambda_k}.$$

For $i \in \{1, \dots, k\}$, $\mathcal{D}_p^{\lambda_i}$ is a T -invariant subspace of T_pM . We note that $\mathcal{D}_p^0 = \text{Ker}T_p$ and $\mathcal{D}_p^1 = \text{Ker}N_p$. \mathcal{D}_p^0 is the maximal anti F -invariant subspace of T_pM where as \mathcal{D}_p^1 is the maximal F -invariant subspace of T_pM . We denote the distributions \mathcal{D}^0 and \mathcal{D}^1 by \mathcal{D}^\perp and \mathcal{D}^T , respectively from now on.

Definition 3.1. ([16]) Let M be a submanifold of a l.p.R. manifold \bar{M} . Then M is said to be a *generic submanifold* if there exists an integer k and functions $\lambda_i, i \in \{1, \dots, k\}$ defined on M with values in $(0, 1)$ such that

- (i) Each $\lambda_i^2(p), i \in \{1, \dots, k\}$ is a distinct eigenvalue of T_p^2 with

$$T_pM = \mathcal{D}_p^\perp \oplus \mathcal{D}_p^T \oplus \mathcal{D}_p^{\lambda_1} \oplus \dots \oplus \mathcal{D}_p^{\lambda_k}$$

for $p \in M$.

- (ii) The dimension of \mathcal{D}^\perp , \mathcal{D}^T and $\mathcal{D}^{\lambda_i}, 1 \leq i \leq k$ are independent of $p \in M$. Moreover, if each λ_i is constant on M , then we say that M is a *skew semi-invariant submanifold* of \bar{M} .

In view of Definition 3.1, we observe that the following special cases.

Let M be a skew semi-invariant submanifold of a l.p.R. manifold \bar{M} as in Definition 3.1. Then

- (a) If $k=0$ and $\mathcal{D}^\perp = \{0\}$, then M is an invariant submanifold [1].
- (b) If $k=0$ and $\mathcal{D}^T = \{0\}$, then M is an anti-invariant submanifold [1].
- (c) If $k=0$, then M is a semi-invariant submanifold [5].
- (d) If $\mathcal{D}^\perp = \{0\} = \mathcal{D}^T$ and $k=1$, then M is a slant submanifold [20].
- (e) If $\mathcal{D}^\perp = \{0\}, \mathcal{D}^T \neq \{0\}$ and $k=1$, then M is a semi-slant submanifold [20].
- (f) If $\mathcal{D}^T = \{0\}, \mathcal{D}^\perp \neq \{0\}$ and $k=1$, then M is a hemi-slant submanifold [25].
- (g) If $\mathcal{D}^\perp = \{0\} = \mathcal{D}^T$ and $k=2$, then M is a bi-slant submanifold [8].

Definition 3.2. A submanifold M of a l.p.R. manifold \bar{M} is called a *skew semi-invariant submanifold of order 1*, if M is a skew semi-invariant submanifold with $k=1$.

In this case, we have

$$(3.3) \quad TM = \mathcal{D}^\perp \oplus \mathcal{D}^T \oplus \mathcal{D}^\theta ,$$

where $\mathcal{D}^\theta = \mathcal{D}^{\lambda_1}$ and λ_1 is constant. We say that a skew semi-invariant submanifold of order 1 is *proper*, if $\mathcal{D}^\perp \neq \{0\}$ and $\mathcal{D}^T \neq \{0\}$.

A slant submanifold M of a l.p.R. manifold \bar{M} is characterized by

$$(3.4) \quad T^2U = \lambda U$$

such that $\lambda \in [0, 1]$, where $U \in TM$, for details; see [20]. Moreover, if θ is the slant angle of M , then we have $\lambda = \cos^2\theta$. On the other hand, for any slant submanifold M of a l.p.R. manifold \bar{M} , we have

$$(3.5) \quad \begin{aligned} (a) \quad T^2 + tN &= I, & (b) \quad \omega^2 + Nt &= I, \\ (c) \quad NT + \omega N &= 0, & (d) \quad Tt + t\omega &= 0. \end{aligned}$$

For the proof of (3.5); see [25].

Throughout this paper, the letters V, W will denote the vector fields of the anti-invariant distribution \mathcal{D}^\perp , U, Z will denote the vector fields of the slant distribution \mathcal{D}^θ and X, Y will denote the vector fields of the invariant distribution \mathcal{D}^T .

For the further study of skew semi-invariant submanifold of order 1 of a l.p.R. manifold, we need to following lemmas.

Lemma 3.3. *Let M be a proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold \bar{M} . Then we have,*

$$(3.6) \quad g(\nabla_V W, X) = -g(A_{FW}V, FX) ,$$

$$(3.7) \quad g(\nabla_V Z, X) = -\csc^2\theta\{g(A_{NTZ}V, X) + g(A_{NZ}V, FX)\} ,$$

$$(3.8) \quad g(\nabla_Z V, X) = -g(A_{FV}Z, FX) ,$$

for $V, W \in \mathcal{D}^\perp, Z \in \mathcal{D}^\theta$ and $X \in \mathcal{D}^T$.

Proof. Using (2.2) and (2.1), we have $g(\nabla_V W, X) = g(\bar{\nabla}_V FW, FX)$ for $V, W \in \mathcal{D}^\perp$ and $X \in \mathcal{D}^T$. Hence, using (2.3), we get (3.6). In a similar way, we have $g(\nabla_V Z, X) = g(\bar{\nabla}_V FZ, FX)$, where $Z \in \mathcal{D}^\theta$. Then using (3.1) and (2.1), we obtain $g(\nabla_V Z, X) = g(\bar{\nabla}_V FTZ, X) + g(\bar{\nabla}_V NZ, FX)$. Hence, using (3.1) and (2.3), we arrive at $g(\nabla_V Z, X) = g(\bar{\nabla}_V T^2Z, X) + g(\bar{\nabla}_V N(TZ), FX) - g(A_{NZ}V, FX)$. With the help of (3.4), (2.2) and (2.3), we get (3.7). Similarly, one can obtain (3.8). \square

Lemma 3.4. *Let M be a proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold \bar{M} . Then we have,*

$$(3.9) \quad g(\nabla_U Z, X) = -\csc^2\theta\{g(A_{NTZ}U, X) + g(A_{NZ}U, FX)\} ,$$

$$(3.10) \quad g(\nabla_X Y, Z) = \csc^2\theta\{g(A_{NTZ}X, Y) + g(A_{NZ}X, FY)\} ,$$

$$(3.11) \quad g(\nabla_X Y, V) = g(A_{FV}X, FY) ,$$

for $X, Y \in \mathcal{D}^T$, $U, Z \in \mathcal{D}^\theta$ and $V \in \mathcal{D}^\perp$.

Proof. Let $U, Z \in \mathcal{D}^\theta$ and $X \in \mathcal{D}^T$. Then using (2.2), (2.1) and (3.1), we have $g(\nabla_U Z, X) = g(\overline{\nabla}_U FZ, FX) = g(\overline{\nabla}_U TZ, FX) + g(\overline{\nabla}_U NZ, FX)$. Again, using (2.1) and (2.3), we obtain $g(\nabla_U Z, X) = g(\overline{\nabla}_U FTZ, X) - g(A_{NZ}U, FX)$. Here, if we use (3.5)-(a) and (3.4), then we get $g(\nabla_U Z, X) = \cos^2\theta g(\nabla_U Z, X) + g(\overline{\nabla}_U NTZ, X) - g(A_{NZ}U, FX)$. After some calculation, we find (3.9). For the proof of (3.10), using (2.2), (2.1) and (3.1), we have $g(\nabla_X Y, Z) = g(\overline{\nabla}_X FY, FZ) = g(\overline{\nabla}_X FY, TZ) + g(\overline{\nabla}_X FY, NZ)$ for $X, Y \in \mathcal{D}^T$ and $Z \in \mathcal{D}^\theta$. Again, using (2.1) and (2.3), we obtain $g(\nabla_X Y, Z) = g(\overline{\nabla}_X Y, FTZ) + g(h(X, FY), NZ)$. With the help of (3.5)-(a) and (3.4), we get $g(\nabla_X Y, Z) = \cos^2\theta g(\nabla_X Y, Z) + g(\overline{\nabla}_X Y, NTZ) + g(h(X, FY), NZ)$. Upon direct calculation, we find (3.10). In a similar way, we can obtain (3.11). \square

Lemma 3.5. *Let M be a proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold \bar{M} . Then we have,*

$$(3.12) \quad g(\nabla_V X, Z) = \csc^2\theta \{g(A_{NTZ}V, X) + g(A_{NZ}V, FX)\} ,$$

$$(3.13) \quad g(\nabla_U Z, V) = \sec^2\theta \{g(A_{FV}U, TZ) + g(A_{NTZ}U, V)\} ,$$

$$(3.14) \quad g(\nabla_X V, Z) = \sec^2\theta \{g(A_{FV}X, TZ) + g(A_{NTZ}X, V)\} ,$$

for $X \in \mathcal{D}^T$, $U, Z \in \mathcal{D}^\theta$ and $V \in \mathcal{D}^\perp$.

Proof. Using (2.2), (2.1) and (3.1), we have $g(\nabla_V X, Z) = g(\overline{\nabla}_V FX, FZ) = -g(\overline{\nabla}_V FZ, FX) = -g(\overline{\nabla}_V TZ, FX) - g(\overline{\nabla}_V NZ, FX)$. Again, using (2.1) and (2.3), we obtain $g(\nabla_V X, Z) = -g(\overline{\nabla}_V FTZ, X) + g(A_{NZ}V, FX)$. Here, using (3.1) and (3.5)-(a), we get $g(\nabla_V X, Z) = -\cos^2\theta g(\nabla_V X, Z) - g(\overline{\nabla}_V NTZ, X) + g(A_{NZ}V, FX)$. According to direct calculation, we arrive at $g(\nabla_V X, Z) = \cos^2\theta g(\nabla_V X, Z) + g(A_{NTZ}V, X) + g(A_{NZ}V, FX)$ which gives (3.12). On the other hand, for any $U, Z \in \mathcal{D}^\theta$ and $V \in \mathcal{D}^\perp$, using (2.2), (2.1) and (3.1), we have $g(\nabla_U Z, V) = g(\overline{\nabla}_U TZ, FV) + g(\overline{\nabla}_U NZ, FV)$. Hence, using (2.2) and (2.1), we obtain $g(\nabla_U Z, V) = g(h(U, TZ), FV) + g(\overline{\nabla}_U FNZ, V)$. Here, if we use (3.2) and (2.4), we get $g(\nabla_U Z, V) = g(A_{FV}U, TZ) + g(\overline{\nabla}_U tNZ, V) + g(\overline{\nabla}_U \omega NZ, V)$. With the help of (3.5)-(a), (3.5)-(c), (3.4) and (2.3), we arrive at $g(\nabla_U Z, V) = g(A_{FV}U, TZ) + g(\overline{\nabla}_U (1 - \cos^2\theta)Z, V) + g(A_{NTZ}U, V)$. Upon direct calculation, we find (3.13). Similarly, we can obtain (3.14). \square

4. WARPED PRODUCT SKEW SEMI-INVARIANT SUBMANIFOLDS OF ORDER 1 OF A LOCALLY PRODUCT RIEMANNIAN MANIFOLD

In this section, we consider a warped product submanifold of type $M = M_1 \times_f M_T$ in a l.p.R. manifold \bar{M} , where M_1 is a hemi-slant submanifold and M_T is an invariant submanifold. Then, it is clear that M is a proper skew semi-invariant submanifold of order 1 of \bar{M} . Thus, from definition of hemi-slant submanifold and skew semi-invariant submanifold of order 1, we have

$$(4.1) \quad TM = \mathcal{D}^\theta \oplus \mathcal{D}^\perp \oplus \mathcal{D}^T .$$

In particular, if $\mathcal{D}^\theta = \{0\}$, then M is a warped product semi-invariant submanifold [21]. If $\mathcal{D}^\perp = \{0\}$, then M is a warped product semi-slant submanifold [22].

Remark 4.1. From Theorem 3.1 of [21], we know that there are no proper warped product semi-invariant submanifolds of type $M_T \times_f M_\perp$ of a l.p.R. manifold \bar{M} such that M_T is invariant submanifold and M_\perp is anti-invariant submanifold of \bar{M} . On the other hand, from Theorem 3.1 of [22], we know that there is no proper warped product submanifold in the form $M_T \times_f M_\theta$ of a l.p.R. manifold \bar{M} such that M_θ is a proper slant submanifold and M_T is an invariant submanifold of \bar{M} . Thus, we conclude that there is no warped product skew semi-invariant submanifold of order 1 of type $M_T \times_f M_1$ of a l.p.R. manifold \bar{M} such that M_1 is a hemi-slant submanifold and M_T is an invariant submanifold of \bar{M} .

We now present an example of warped product semi-invariant submanifold of order 1 of type $M_1 \times_f M_T$ in a l.p.R. manifold.

Example 4.2. Consider the locally product Riemannian manifold $\mathbb{R}^{10} = \mathbb{R}^5 \times \mathbb{R}^5$ with usual metric g and almost product structure F defined by

$$F(\partial_i) = \partial_i, \quad F(\partial_j) = -\partial_j, \quad$$

where $i \in \{1, \dots, 5\}, j \in \{6, \dots, 10\}, \partial_k = \frac{\partial}{\partial x_k}$ and (x_1, \dots, x_{10}) are natural coordinates of \mathbb{R}^{10} . Let M be a submanifold of $\bar{M} = (\mathbb{R}^{10}, g, F)$ given by

$$\phi(x, y, z, u, v) = (x + y, x - y, x \cos u, x \sin u, z, -z, x, \frac{2}{\sqrt{3}}y, x \cos v, x \sin v),$$

where $x > 0$.

Then, we easily see that the local frame of TM is spanned by

$$\begin{aligned} \phi_x &= \partial_1 + \partial_2 + \cos u \partial_3 + \sin u \partial_4 + \partial_7 + \cos v \partial_9 + \sin v \partial_{10}, \\ \phi_y &= \partial_1 - \partial_2 + \frac{2}{\sqrt{3}} \partial_8, \quad \phi_z = \partial_5 - \partial_6, \\ \phi_u &= -x \sin u \partial_3 + x \cos u \partial_4, \quad \phi_v = -x \sin v \partial_9 + x \cos v \partial_{10}. \end{aligned}$$

Then by direct calculation, we see that $\mathcal{D}^\theta = \text{span}\{\phi_x, \phi_y\}$ is a slant distribution with slant angle $\theta = \arccos \frac{1}{5}$ and $\mathcal{D}^\perp = \text{span}\{\phi_z\}$ is an anti-invariant distribution since $F(\phi_z)$ is orthogonal to TM . Moreover, $\mathcal{D}^T = \text{span}\{\phi_u, \phi_v\}$ is an invariant distribution. Thus, we conclude that M is a proper skew semi-invariant submanifold of order 1 of \bar{M} . Furthermore, one can easily see that $\mathcal{D}^\theta \oplus \mathcal{D}^\perp$ and \mathcal{D}^T are integrable. If we denote the integral submanifolds $\mathcal{D}^\theta, \mathcal{D}^\perp$ and \mathcal{D}^T by M_θ, M_\perp and M_T , respectively, then the induced metric tensor of M is

$$\begin{aligned} ds^2 &= 5dx^2 + \frac{10}{3}dy^2 + 2dz^2 + x^2(du^2 + dv^2) \\ &= g_{M_\theta} + g_{M_\perp} + x^2 g_{M_T}. \end{aligned}$$

Thus, $M = (M_\theta \times M_\perp) \times_{x^2} M_T$ is a warped product skew semi-invariant submanifold of order 1 of \bar{M} with warping function $f = x$.

Let \mathcal{D}^θ and \mathcal{D}^T be slant and invariant distributions on M , respectively. Then we say that M is $(\mathcal{D}^\theta, \mathcal{D}^T)$ *mixed totally geodesic* if $h(Z, X) = 0$, where $Z \in \mathcal{D}^\theta$ and $X \in \mathcal{D}^T$ [19].

Before giving a necessary and sufficient condition for skew semi-invariant submanifold of order 1 to be a locally warped product, we recall that the S. Hiepko's result [13], (cf. [12], Remark 2.1): Let \mathcal{D}_1 be a vector subbundle in the tangent bundle of a Riemannian manifold M and let \mathcal{D}_2 be its normal bundle. Suppose that the two distributions are involutive. If we denote by M_1 and M_2 the integral

manifolds of \mathcal{D}_1 and \mathcal{D}_2 , respectively, then M is locally isometric to warped product $M_1 \times_f M_2$ if the integral manifold M_1 totally geodesic and the integral manifold M_2 is an extrinsic sphere, in other word, M_2 is a totally umbilical submanifold with a parallel mean curvature vector.

Theorem 4.3. *Let M be a $(\mathcal{D}^\theta, \mathcal{D}^T)$ mixed totally geodesic proper skew semi-invariant submanifold of order 1 with integrable distribution \mathcal{D}^T of a l.p.R. manifold \bar{M} . Then M is a locally warped product submanifold if and only if*

$$(4.2) \quad A_{FV}FX = -V[\sigma]X \ ,$$

and

$$(4.3) \quad A_{NZ}FX + A_{NTZ}X = -Z[\sigma] \sin^2 \theta X$$

for $X \in \mathcal{D}^T$, $Z \in \mathcal{D}^\theta$, $V \in \mathcal{D}^\perp$ and a function σ defined on M such that $Y[\sigma] = 0$ for $Y \in \mathcal{D}^T$.

Proof. Let $M = M_1 \times_f M_T$ be a $(\mathcal{D}^\theta, \mathcal{D}^T)$ mixed totally geodesic warped product proper skew semi-invariant submanifold of order 1 with integrable distribution \mathcal{D}^T of a l.p.R. manifold \bar{M} . Then using (3.6) and (3.8), we have $g(A_{FV}W, FX) = 0$ and $g(A_{FV}Z, FX) = 0$ for any $V, W \in \mathcal{D}^\perp$, $Z \in \mathcal{D}^\theta$ and $X \in \mathcal{D}^T$. Since A is self adjoint, we deduce that $A_{FV}FX$ has no components in TM_1 . So $A_{FV}FX \in \mathcal{D}^T$. Thus, using (2.2), (2.1) and (1.1), for any $Y \in \mathcal{D}^T$, we obtain $g(A_{FV}FX, Y) = -g(\bar{\nabla}_Y FV, FX) = -g(\bar{\nabla}_Y V, X) = -g(\nabla_Y V, X) = -V(\ln f)g(X, Y)$. Which proves (4.2). Since M is $(\mathcal{D}^\theta, \mathcal{D}^T)$ mixed totally geodesic for any $Z \in \mathcal{D}^\theta$ and $X \in \mathcal{D}^T$, we have $g(A_{NTZ}X, Z) = 0$. It means that $A_{NTZ}X$ has no components in \mathcal{D}^θ . On the other hand, from Lemma 3.3 of [25], we know that $TZ \in \mathcal{D}^\theta$ for any $Z \in \mathcal{D}^\theta$. Thus, using this fact and (1.1), from (3.14), we get $g(A_{NTZ}X, V) = 0$, that is, $A_{NTZ}X$ has no components in \mathcal{D}^\perp . Thus, from (4.1), we conclude that $A_{NTZ}X \in \mathcal{D}^T$. Also, we have $A_{NZ}X \in \mathcal{D}^T$. Then, for $X, Y \in \mathcal{D}^T$ and $Z \in \mathcal{D}^\theta$, with the help of (1.1), from (3.10), we have $g(A_{NTZ}Y, X) + g(A_{NZ}FY, X) = -\sin^2 \theta g(\nabla_X Z, Y) = -\sin^2 \theta Z(\ln f)g(Y, X)$. This proves (4.3). Moreover, $Y(\ln f) = 0$ for a warped product proper skew semi-invariant submanifold of order 1, we obtain $\sigma = \ln f$.

Conversely, suppose that M is $(\mathcal{D}^\theta, \mathcal{D}^T)$ mixed totally geodesic proper skew semi-invariant submanifold of order 1 with integrable distribution \mathcal{D}^T of a l.p.R. manifold \bar{M} such that (4.2) and (4.3) hold. We know from Theorem 4.6 of [25], \mathcal{D}^\perp is always integrable. So, we have $g(\nabla_V W, X) = 0$ for $V, W \in \mathcal{D}^\perp$ and $X \in \mathcal{D}^T$. Using this fact, (4.2), (4.3) and (3.7)-(3.9), it is not difficult to see that M_1 is totally geodesic in M . Let M_T be the integral manifold of \mathcal{D}^T and h_T be the second fundamental form of M_T in M . From (2.2), we have $g(h_T(X, Y), V) = g(\nabla_X Y, V)$ for $X, Y \in \mathcal{D}^T$ and $V \in \mathcal{D}^\perp$. Then, (3.11) imply that $g(h_T(X, Y), V) = g(A_{FV}FY, X)$. Thus, using (4.2), we obtain

$$(4.4) \quad g(h_T(X, Y), V) = -V[\sigma]g(Y, X) \ .$$

Similarly, from (2.2), we have $g(h_T(X, Y), Z) = g(\nabla_X Y, Z)$ for $X, Y \in \mathcal{D}^T$ and $Z \in \mathcal{D}^\theta$. Using (3.10), we obtain $g(h_T(X, Y), Z) = \csc^2 \theta \{g(A_{NTZ}Y, X) + g(A_{NZ}FY, X)\}$. Thus, from (4.3), we get

$$(4.5) \quad g(h_T(X, Y), Z) = -Z[\sigma]g(X, Y) \ .$$

Thus, for any $E = V + Z \in TM_1$, from (4.4) and (4.5), we arrive at

$$(4.6) \quad \begin{aligned} g(h_T(X, Y), E) &= g(h_T(X, Y), V) + g(h_T(X, Y), Z) \\ &= -\{V[\sigma] + Z[\sigma]\}g(X, Y). \end{aligned}$$

Last equation (4.6) says that M_T is totally umbilical in M . Let denote by $grad^\perp \sigma$ and $grad^\theta \sigma$ the gradient of σ on \mathcal{D}^\perp and \mathcal{D}^θ , respectively. From (4.6), we write

$$(4.7) \quad h_T(X, Y) = -\{grad^\perp \sigma + grad^\theta \sigma\}g(X, Y).$$

Thus, for any $E = V + Z \in TM_1$, we have

$$\begin{aligned} g(\nabla_X(grad^\perp \sigma + grad^\theta \sigma), E) &= g(\nabla_X grad^\perp \sigma, E) + g(\nabla_X grad^\theta \sigma, E) \\ &= \{Xg(grad^\perp \sigma, V) - g(grad^\perp \sigma, \nabla_X E)\} \\ &\quad + \{Xg(grad^\theta \sigma, Z) - g(grad^\theta \sigma, \nabla_X E)\} \\ &= X[V[\sigma]] - g(grad^\perp \sigma, \nabla_X E) + X[Z[\sigma]] - g(grad^\theta \sigma, \nabla_X E). \end{aligned}$$

On the other hand, if we use (4.3) in (3.14), then we get $g(\nabla_X V, Z) = -g(\nabla_X Z, V) = 0$. Using this fact, we obtain

$$g(\nabla_X(grad^\perp \sigma + grad^\theta \sigma), E) = X[V[\sigma]] - g(grad^\perp \sigma, \nabla_X Z) + X[Z[\sigma]] - g(grad^\theta \sigma, \nabla_X V)$$

Upon direct calculation, we arrive at

$$\begin{aligned} g(\nabla_X(grad^\perp \sigma + grad^\theta \sigma), E) &= \{X[Z[\sigma]] - [X, Z][\sigma] + g(grad^\perp \sigma, \nabla_Z X)\} \\ &\quad + \{X[V[\sigma]] - [X, V][\sigma] + g(grad^\theta \sigma, \nabla_V X)\}. \end{aligned}$$

After some calculation, we get

$$\begin{aligned} g(\nabla_X(grad^\perp \sigma + grad^\theta \sigma), E) &= \{Z[X[\sigma]] + g(grad^\perp \sigma, \nabla_Z X) + V[X[\sigma]] + g(grad^\theta \sigma, \nabla_V X)\}. \end{aligned}$$

Since $X[\sigma] = 0$, from the last equation, we derive

$$g(\nabla_X(grad^\perp \sigma + grad^\theta \sigma), E) = -g(\nabla_Z grad^\perp \sigma, X) - g(\nabla_V grad^\theta \sigma, X).$$

Here, we know that $\nabla_Z grad^\perp \sigma, \nabla_V grad^\theta \sigma \in TM_1$, since M_1 is totally geodesic. Hence, we obtain $g(\nabla_X(grad^\perp \sigma + grad^\theta \sigma), E) = 0$. It means that $grad^\perp \sigma + grad^\theta \sigma$ is parallel in M . This fact and (4.7) imply that M_T is an extrinsic sphere. This completes the proof. \square

5. A CHEN-TYPE INEQUALITY FOR WARPED PRODUCT SKEW SEMI-INVARIANT SUBMANIFOLDS OF ORDER 1

In this section, we prove that the invariant distribution which is involved in the definition of the warped product proper skew semi-invariant submanifolds of order 1 of a l.p.R. manifold is integrable under some restrictions. We also give an inequality similar to Chen's inequality [10] for the squared norm of the second fundamental form in terms of the warping function for such submanifolds. We first give the following two lemmas for later use.

Lemma 5.1. *Let $M = M_1 \times_f M_T$ be a warped product proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold \bar{M} . Then we have,*

$$(5.1) \quad g(h(X, V), FW) = 0$$

and

$$(5.2) \quad g(h(X, V), NZ) = 0,$$

for $X \in \mathcal{D}^T$, $Z \in \mathcal{D}^\theta$ and $V, W \in \mathcal{D}^\perp$.

Proof. For any $V, W \in \mathcal{D}^\perp$ and $X \in \mathcal{D}^T$, using (2.2), (2.1) and (1.1), we get $g(h(X, V), FW) = g(\bar{\nabla}_V X, FW) = g(\bar{\nabla}_V FX, W) = g(\nabla_V FX, W) = V(\ln f)g(FX, W) = 0$, since $g(FX, W) = 0$. Hence (5.1) follows. In a similar way, using (2.2), (2.1), (3.1) and (1.1), we have

$$\begin{aligned} g(h(X, V), NZ) &= g(\bar{\nabla}_V X, NZ) = g(\bar{\nabla}_V X, FZ) - g(\bar{\nabla}_V X, TZ) \\ &= g(\bar{\nabla}_V FX, Z) - g(\bar{\nabla}_V X, TZ) \\ &= g(\nabla_V FX, Z) - g(\nabla_V X, TZ) \\ &= V(\ln f)g(FX, Z) - V(\ln f)g(X, TZ) = 0, \end{aligned}$$

since $g(FX, Z) = 0$ and $g(X, TZ) = 0$. \square

Lemma 5.2. *Let $M = M_1 \times_f M_T$ be a warped product proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold \bar{M} . Then we have,*

$$(5.3) \quad g(h(X, FY), FV) = -V(\ln f)g(X, Y)$$

and

$$(5.4) \quad g(h(X, Y), NZ) = TZ(\ln f)g(X, Y)$$

for $X, Y \in \mathcal{D}^T$, $Z \in \mathcal{D}^\theta$ and $V \in \mathcal{D}^\perp$.

Proof. Using (2.2) and (2.1), we have

$g(h(X, FY), FV) = g(\bar{\nabla}_X FY, FV) = g(\bar{\nabla}_X Y, V) = g(\nabla_X Y, V) = -g(\nabla_X V, Y)$ for any $X, Y \in \mathcal{D}^T$ and $V \in \mathcal{D}^\perp$. Hence, using (1.1), we get easily (5.3). Last assertion (5.4) follows from Lemma 3.1-(ii) of [2] by using linearity. \square

Theorem 5.3. *Let $M = M_1 \times_f M_T$ be an $(q + m)$ -dimensional warped product proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold \bar{M} of dimension $2q + m$, where $\dim(M_1) = q$ and $\dim(M_T) = m$. Then the invariant distribution \mathcal{D}^T of M_T is integrable.*

Proof. For any $X, Y \in \mathcal{D}^T$, $Z \in \mathcal{D}^\theta$ and $V \in \mathcal{D}^\perp$, using (5.3) and (5.4), we get $g(h(X, FY), FV) = g(h(FX, Y), FV)$ and $g(h(X, FY), NZ) = g(h(FX, Y), NZ)$, since $g(X, FY) = g(FX, Y)$. Hence, we conclude that $h(X, FY) = h(FX, Y)$, since $T^\perp M = F\mathcal{D}^\perp \oplus N\mathcal{D}^\theta$, where $T^\perp M$ is the normal bundle of M in \bar{M} . Thus, our assertion immediately comes from Theorem 1 of [5]. \square

Let M be a $(k + n + m)$ -dimensional warped product proper skew semi-invariant submanifold of order 1 of a $(2k + 2n + m)$ -dimensional l.p.R. manifold \bar{M} . We choose a canonical orthonormal basis $\{e_1, \dots, e_m, \bar{e}_1, \dots, \bar{e}_k, \tilde{e}_1, \dots, \tilde{e}_n, e_1^*, \dots, e_k^*, F\bar{e}_1, \dots, F\bar{e}_n\}$ such that $\{e_1, \dots, e_m\}$ is an orthonormal basis of \mathcal{D}^T , $\{\bar{e}_1, \dots, \bar{e}_k\}$ is an orthonormal basis of \mathcal{D}^θ , $\{\tilde{e}_1, \dots, \tilde{e}_n\}$ is an orthonormal basis of \mathcal{D}^\perp , $\{e_1^*, \dots, e_k^*\}$ is an orthonormal basis of $N\mathcal{D}^\theta$ and $\{F\bar{e}_1, \dots, F\bar{e}_n\}$ is an orthonormal basis of $F\mathcal{D}^\perp$.

Remark 5.4. In view of (2.1), we can observe that $\{Fe_1, \dots, Fe_m\}$ is also an orthonormal basis of \mathcal{D}^T . On the other hand, with the help of the equations (3.5) and (3.6) of [25], we can see that $\{\sec\theta T\bar{e}_1, \dots, \sec\theta T\bar{e}_k\}$ is also an orthonormal basis of \mathcal{D}^θ and $\{\csc\theta N\bar{e}_1, \dots, \csc\theta N\bar{e}_k\}$ is also an orthonormal basis of $N\mathcal{D}^\theta$.

We now state the main result of this section.

Theorem 5.5. *Let $M = M_1 \times_f M_T$ be a $(k + n + m)$ -dimensional warped product proper skew semi-invariant submanifold of order 1 of a $(2k + 2n + m)$ -dimensional l.p.R. manifold \bar{M} . Then the squared norm of the second fundamental form of M satisfies*

$$(5.5) \quad \|h\|^2 \geq m\{\|\nabla^\perp(\ln f)\|^2 + \cot^2\theta\|\nabla^\theta(\ln f)\|^2\},$$

where $m = \dim(M_T)$, $\nabla^\perp(\ln f)$ and $\nabla^\theta(\ln f)$ are gradients of $\ln f$ on \mathcal{D}^\perp and \mathcal{D}^θ , respectively. If the equality case of (5.5) holds, then M_1 is a totally geodesic submanifold of \bar{M} and M is mixed totally geodesic. Moreover, M_T can not be minimal.

Proof. In view of decomposition (4.1), the squared norm of the second fundamental form h can be decomposed as

$$\begin{aligned} \|h\|^2 &= \|h(\mathcal{D}^T, \mathcal{D}^T)\|^2 + \|h(\mathcal{D}^\theta, \mathcal{D}^\theta)\|^2 + \|h(\mathcal{D}^\perp, \mathcal{D}^\perp)\|^2 \\ &\quad + 2\|h(\mathcal{D}^T, \mathcal{D}^\perp)\|^2 + 2\|h(\mathcal{D}^T, \mathcal{D}^\theta)\|^2 + 2\|h(\mathcal{D}^\perp, \mathcal{D}^\theta)\|^2. \end{aligned}$$

Which can be written as follows:

$$\begin{aligned} (5.6) \quad \|h\|^2 &= \sum_{i,j=1}^m \sum_{a=1}^n g(h(e_i, e_j), F\tilde{e}_a)^2 + \sum_{i,j=1}^m \sum_{r=1}^k g(h(e_i, e_j), e_r^*)^2 \\ &\quad + \sum_{a,b,c=1}^n g(h(\tilde{e}_a, \tilde{e}_b), F\tilde{e}_c)^2 + \sum_{a,b=1}^n \sum_{r=1}^k g(h(\tilde{e}_a, \tilde{e}_b), e_r^*)^2 \\ &\quad + \sum_{r,s=1}^k \sum_{a=1}^n g(h(\bar{e}_r, \bar{e}_s), F\tilde{e}_a)^2 + \sum_{r,s,q=1}^k g(h(\bar{e}_r, \bar{e}_s), e_q^*)^2 \\ &\quad + 2 \sum_{i=1}^m \sum_{a,b=1}^n g(h(e_i, \tilde{e}_a), F\tilde{e}_b)^2 + 2 \sum_{i=1}^m \sum_{a=1}^n \sum_{r=1}^k g(h(e_i, \tilde{e}_a), e_r^*)^2 \\ &\quad + 2 \sum_{i=1}^m \sum_{r=1}^k \sum_{a=1}^n g(h(e_i, \bar{e}_r), F\tilde{e}_a)^2 + 2 \sum_{i=1}^m \sum_{r,s=1}^k g(h(e_i, \bar{e}_r), e_s^*)^2 \\ &\quad + 2 \sum_{r=1}^k \sum_{a,b=1}^n g(h(\bar{e}_r, \tilde{e}_a), F\tilde{e}_b)^2 + 2 \sum_{r,s=1}^k \sum_{a=1}^n g(h(\bar{e}_r, \tilde{e}_a), e_s^*)^2. \end{aligned}$$

Here, using (5.1)-(5.3) and Remark 5.4, we have

$$(5.7) \quad \sum_{i,j=1}^m \sum_{a=1}^n g(h(e_i, e_j), F\tilde{e}_a)^2 = \sum_{i,j=1}^m \sum_{a=1}^n (-\tilde{e}_a(\ln f)g(e_i, e_j))^2$$

and

$$(5.8) \quad \sum_{i,j=1}^m \sum_{r=1}^k g(h(e_i, e_j), e_r^*)^2 = \sum_{i,j=1}^m \sum_{r=1}^k g(h(e_i, e_j), N\bar{e}_r)^2 \csc^2\theta.$$

Also, using (5.4) from (5.8), we get

$$(5.9) \quad \sum_{i,j=1}^m \sum_{r=1}^k g(h(e_i, e_j), e_r^*)^2 = \sum_{i,j=1}^m \sum_{r=1}^k (T\bar{e}_r(\ln f)g(e_i, e_j))^2 \csc^2 \theta.$$

Using (5.7) and (5.9) from (5.6), we get

$$(5.10) \quad \|h\|^2 \geq m\|\nabla^\perp(\ln f)\|^2 + \sum_{i,j=1}^m \sum_{r=1}^k (T\bar{e}_r(\ln f)g(e_i, e_j))^2 \csc^2 \theta.$$

In view of Remark 5.4, we replace \bar{e}_r by $\sec\theta T\bar{e}_r$ in the last term of (5.10) and using (3.4), we have

$$(5.11) \quad \begin{aligned} & \sum_{i,j=1}^m \sum_{r=1}^k (T\bar{e}_r(\ln f)g(e_i, e_j))^2 \csc^2 \theta \\ &= \sum_{i,j=1}^m \sum_{r=1}^k \cos^4 \theta (\bar{e}_r(\ln f)g(e_i, e_j))^2 \csc^2 \theta = m \cot^2 \theta \|\nabla^\theta(\ln f)\|^2. \end{aligned}$$

Thus, using (5.11) in (5.10), we find (5.5).

Next, if the equality case of (5.5) holds, then from (5.6), we have

$$(5.12) \quad h(\mathcal{D}^\perp, \mathcal{D}^\perp) = 0, \quad h(\mathcal{D}^\theta, \mathcal{D}^\theta) = 0, \quad h(\mathcal{D}^\perp, \mathcal{D}^\theta) = 0$$

and

$$(5.13) \quad h(\mathcal{D}^T, \mathcal{D}^\perp) = 0, \quad h(\mathcal{D}^T, \mathcal{D}^\theta) = 0.$$

Since M_1 is totally geodesic in M , from (5.12) it follows that M_1 is also totally geodesic in \bar{M} . On the other hand (5.13) imply that M is mixed totally geodesic. Finally, if we suppose that M is minimal, then from (5.3) and (5.4), we conclude that $\|\nabla(\ln f)\| = 0$, which is a contradiction. \square

Remark 5.6. Theorem 5.5 coincides with Theorem 4.2 of [21] in case $\mathcal{D}^\theta = \{0\}$. In other word, Theorem 5.5 is a generalization of Theorem 4.2 of [21].

REFERENCES

1. T. Adati, Submanifolds of an almost product manifold, *Kodai Math. J.* **4** (1981), no. 2, 327–343.
2. F.R. Al-Solamy and M.A. Khan, Warped product submanifolds of Riemannian product manifolds, *Abstract and Applied Analysis* (2012), Article ID 724898, 12 pages, doi:10.1155/2012/724898.
3. K. Arslan, A. Carriazo, B. Y. Chen and C. Murathan, On slant submanifolds of neutral Kaehler manifolds, *Taiwanese J. Math.* **17** (2010), no. 2, 561–584.
4. A. Bejancu, CR-Submanifolds of Kaehler manifold I, *Proc. Amer. Math. Soc.* **69** (1978), 135–142.
5. A. Bejancu, Semi-invariant submanifolds of locally product Riemannian manifolds, *An. Univ. Timișoara Ser. Științ. Math. Al.* **22** (1984), no. 1-2, 3–11.
6. R.L. Bishop and B. O'Neill, Manifolds of negative curvature, *Trans. Amer. Math. Soc.* **145** (1969), 1–49.
7. J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez, Slant submanifolds in Sasakian manifolds, *Glasgow Math. J.*, **42** (2000), 125–138.
8. A. Carriazo, *Bi-slant immersions*, in: Proc. ICRAMS 2000, Kharagpur, India, 2000, 88–97.
9. B.Y. Chen, Slant immersions, *Bull. Austral. Math. Soc.*, **41** (1990), no. 1, 135–147.
10. B.Y. Chen, Geometry of warped product in Kaehler manifolds, *Monatsh. Math.* **133** (2001), 177–195.

11. B.Y. Chen, Geometry of warped product submanifolds: a survey, *J. Adv. Math. Stud.* **6** (2013), 1-43.
12. F. Dillen and S. Nölker, Semi-parallelity multi rotation surfaces and the helix property, *J. Reine Angew Math.* **435** (1993), 33-63.
13. S. Hiepko, Eine innere Kennzeichnung der verzerrten Produkte, *Math. Ann.*, **241** (1979), 209-215.
14. V.A. Khan and M.A. Khan, Pseudo-slant submanifolds of a Sasakian manifold, *Indian J. Pure Appl. Math.*, **38** (2007), 31-42.
15. S.M. Khursheed Haider, M. Thakur, and Advin, Warped product skew CR-submanifolds of a Cosymplectic manifold, *Lobachevskii J. Math.* **33** (2012), no. 3, 262-273.
16. X. Liu and Fang-Ming Shao, Skew semi-invariant submanifolds of a locally product manifold, *Portugaliae Math.* **56** (1999), no. 3, 319-327.
17. A. Lotta, Slant submanifolds in contact geometry, *Bull. Math. Soc. Sci. Roumanie*, **39**(1996), 183-198.
18. N. Papaghiuc, Semi-slant submanifolds of a Kählerian manifold, *Ann. Şt. Al. I. Cuza Univ. Iaşi*, **40** (1994), 55-61.
19. G.S. Ronsse, Generic and skew CR-submanifolds of a Kähler manifold, *Bull. Inst. Math. Acad. Sinica*, **18** (1990), 127-141.
20. B. Şahin, Slant submanifolds of an almost product Riemannian manifold, *J. Korean Math. Soc.* **43** (2006), no. 4, 717-732.
21. B. Şahin, Warped product Semi-invariant submanifolds of a locally product Riemannian manifold, *Bull. Math. Soc. Sci. Roumanie*, **49(97)** (2006), No.4, 383-394.
22. B. Şahin, Warped product Semi-slant submanifolds of a locally product Riemannian manifold, *Studia Sci. Math. Hungarica*, **46** (2009), No.2, 169-184.
23. B. Şahin, Warped product submanifolds of a Kähler manifold with a slant factor, *Ann. Pol. Math.* **95** (2009), no. 3, 207-226.
24. B. Şahin, Skew CR-warped product submanifolds of a Kähler manifolds, *Math. Commun.* **15** (2010), no. 1, 189-204.
25. H.M. Taştan and F. Özdemir, The geometry of hemi-slant submanifolds of a locally product Riemannian manifold, (submitted) (2014), arXiv:1405.6687v1 [math. DG].
26. M.M. Tripathi, Generic submanifolds of generalized complex space forms, *Publ. Math. Debrecen*, **50** (1997), no. 3-4, 373-392.
27. S. Uddin, A.Y.M. Chi, Warped product pseudo-slant submanifolds of nearly Kaehler manifolds, *An. Şt. Univ. Ovidius Constanţa*. **19**(3) (2011), 195-204.
28. K. Yano and M. Kon, *Structures on Manifolds*, World Scientific, Singapore, 1984.

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